Vortex formation in swarms of interacting particles

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(Received 28 June 2006; revised manuscript received 9 November 2006; published 28 March 2007)

Vortexlike swarming behavior is observed in a wide range of biological systems. In the work reported here a discrete particle model is used to investigate the onset of such vortexlike behavior in a swarm of interacting particles. A constrained minimization of the total effective energy of the swarm of particles is performed, with the total angular momentum of the swarm conserved. It is shown that the emergence of vortexlike behavior can then be viewed as a constrained minimum energy configuration which the swarm relaxes into.

DOI: 10.1103/PhysRevE.75.032904 PACS number(s): 87.23.Cc, 87.10.+e, 89.75.Fb

Vortexlike swarming behavior is observed in a wide range of biological systems from bacteria to higher vertebrates [1–3]. Such behavior is also seen in simulations of interacting particles which attempt to capture the processes through which individual agents in a swarm form coherent spatial patterns [4–6]. Previous studies have used artificial interaction potentials to model long-range aggregation and short-range repulsion within a swarm [5,6]. Propagating the motion of the swarm using such artificial potential fields shows that swarms of interacting particles can relax into vortexlike states. In the work reported here, constrained minimization of the total effective energy of the swarm is performed, with the total angular momentum of the swarm conserved. Angular momentum conservation arises from the pair-wise interaction between members of the swarm, while energy minimization arises through dissipation as swarm members interact. Vortexlike spatial patterns can then be viewed as a constrained minimum-energy configuration. The interactions between swarm members are chosen to reflect rule based approaches which have been successful in both modeling [7] and laboratory experimentation [8].

A swarm of biological or artificial agents is comprised of N identical particles of mass m with position and velocity $(\mathbf{x}_i, \mathbf{v}_i)$ defining the state of the ith particle. As will be seen later, the center-of-mass of the swarm C forms an inertial origin relative to the fixed frame O, as shown in Fig. 1. Aggregation of the swarm is achieved through a long-range attractive potential $U_{ij}^a = -C_a \exp(-|\mathbf{x}_{ij}|/l_a)$, while collision between particles is prevented through a short-range repulsive potential $U_{ij}^r = C_r \exp(-|\mathbf{x}_{ij}|/l_r)$ [5,6]. The strength of the attractive and repulsive potential is defined by C_a and C_r with range l_a and l_r, such that $l_a > l_r$. In addition, it will be assumed that particles in the swarm attempt to align with their neighbors locally through a velocity dependent orientation function $\mathbf{\Lambda}_i$, as used in rule based approaches to swarm modeling [7]. Separate propulsive and drag forces acting on each particle in the swarm can be included, however these quickly equilibrate so that the swarm is largely driven by the interaction of the potential fields [6].

The swarm of interacting particles will now be defined through the interaction potential and orientation function such that

$$\dot{\mathbf{x}}_i = \mathbf{v}_i, \tag{1a}$$

$$m\dot{\mathbf{v}}_i = -\nabla U_i^a - \nabla U_i^r - \mathbf{\Lambda}_i, \tag{1b}$$

where $U_i = \Sigma_j U_{ij}$ and $\nabla(\cdot) = \partial(\cdot)/\partial \mathbf{x}_i$. The orientation function can be defined as $\mathbf{\Lambda}_i = \Sigma_j C_o(\mathbf{v}_{ij} \cdot \hat{\mathbf{x}}_{ij}) \exp(-|\mathbf{x}_{ij}|/l_o) \hat{\mathbf{x}}_{ij}$, where $(\hat{\cdot})$ denotes a unit vector, C_o is the strength of the orientation function and l_o is the range over which the orientation interaction occurs. With this function, motion towards or away from neighbors is weakly damped, proportional to the component of relative velocity along the vector connecting neighboring particles. This results in a local alignment of particle velocity vectors, as used in rule based approaches [7]. Later the three competing terms in Eq. (1b) will be defined in a hierarchy such that $l_r < l_o < l_a$. This hierarchy is equivalent to the zone of repulsion, zone of orientation, and zone of attraction used successfully in both simulation [7] and laboratory experimentation [8].

The effective energy of the swarm can be obtained by direct summation of Eq. (1b) so that $\Sigma_i m\mathbf{v}_i \cdot \dot{\mathbf{v}}_i = -\Sigma_i \mathbf{v}_i \cdot \nabla U_i^a - \Sigma_i \mathbf{v}_i \cdot \nabla U_i^r - \Sigma_i \mathbf{v}_i \cdot \mathbf{\Lambda}_i$. The effective energy is then defined though a summation to evaluate each pair-wise potential interaction and a summation of the kinetic energy of each particle. The total kinetic energy T of the swarm is simply $T = 1/2 \Sigma_i m \mathbf{v}_i^2$ and the total effective potential U is determined from $U = 1/2 \Sigma_i \Sigma_j (U_{ij}^a + U_{ij}^r)$. It can then be shown that $d/dt(T+U) = -\Sigma_i \mathbf{v}_i \cdot \mathbf{\Lambda}_i$ so that the swarm of particles will slowly leak energy and relax to a minimum-energy state where $\Sigma_i \mathbf{v}_i \cdot \mathbf{\Lambda}_i = 0$. This condition is satisfied if $\mathbf{v}_{ij} \cdot \hat{\mathbf{x}}_{ij} = 0$, corresponding to a state of relative equilibrium with a fixed separation distance $|\mathbf{x}_{ij}|$ between the particles and local alignment of velocity vectors.

The total angular momentum of the swarm is obtained by a summation of the couple from each pair-wise interaction within the swarm. Summing Eq. (1b) it can be seen that $\Sigma_i m \mathbf{x}_i \times \dot{\mathbf{v}}_i = -\Sigma_i \mathbf{x}_i \times \nabla U_i^a - \Sigma_i \mathbf{x}_i \times \nabla U_i^r - \Sigma_i \mathbf{x}_i \times \mathbf{\Lambda}_i$. However, both the gradient of the potential field and the orientation function are formed by the summation of pair-wise interactions along $\hat{\mathbf{x}}_{ij}$, the vector connecting each pair of particles. A consequence is that the summations on the right will vanish due to the internal symmetry of the interactions within the swarm. Pairs of internal torque couples will cancel as can be seen using the identity $\mathbf{x}_i \times \mathbf{x}_j = -\mathbf{x}_j \times \mathbf{x}_i$ in the summations. Identifying the angular momentum of the ith particle as $\mathbf{L}_i$

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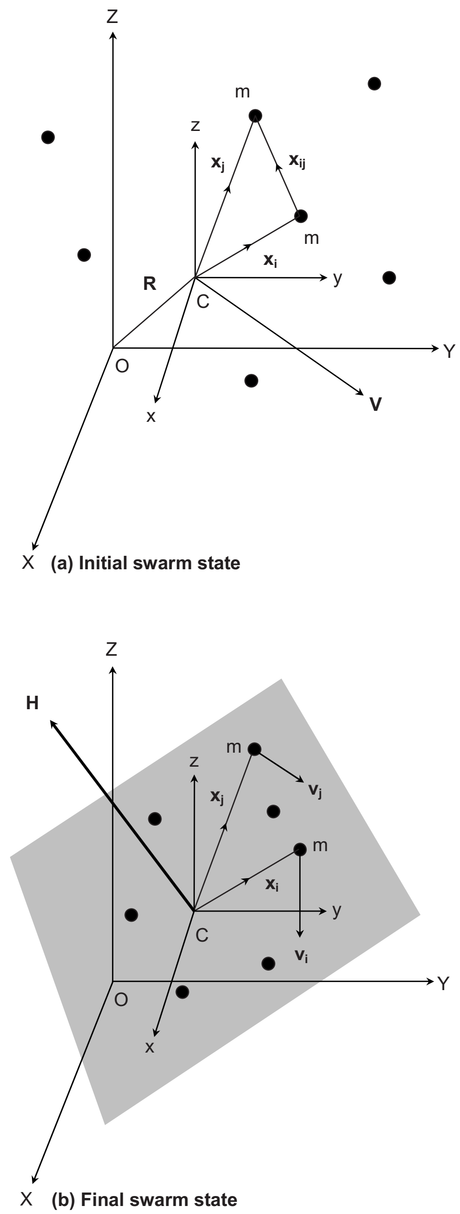

FIG. 1. (a) Swarm of interacting particles with center-of-mass C at position $\mathbf{R}$ relative to an inertial origin O. Due to internal symmetry in the pair-wise interaction between particles, the center-of-mass is inertial and translates with uniform velocity $\mathbf{V}$. (b) Relaxation of the swarm of interacting particles to a vortexlike state with angular momentum $\mathbf{H}$. The swarm rotates with rigid body motion about the vector $\mathbf{H}$ in a minimum energy configuration.

$=m\mathbf{x}_i \times \mathbf{v}_i$ it can be seen that $\Sigma_i d\mathbf{L}_i/dt=0$ and so the total angular momentum of the swarm $\mathbf{H}=\Sigma_i \mathbf{L}_i$ is conserved. Through a similar argument it can be shown that the total linear momentum of the swarm is also conserved. The center-of-mass of the swarm C at position $\mathbf{R}$ therefore translates with constant velocity $\mathbf{V}$ and so forms an inertial frame of reference.

It has been shown that the swarm of particles dissipates energy, but that the total angular momentum of the swarm is conserved as it relaxes. Possible states of the swarm can be explored by using a Lagrange multiplier to enforce conservation of angular momentum, while energy slowly leaks

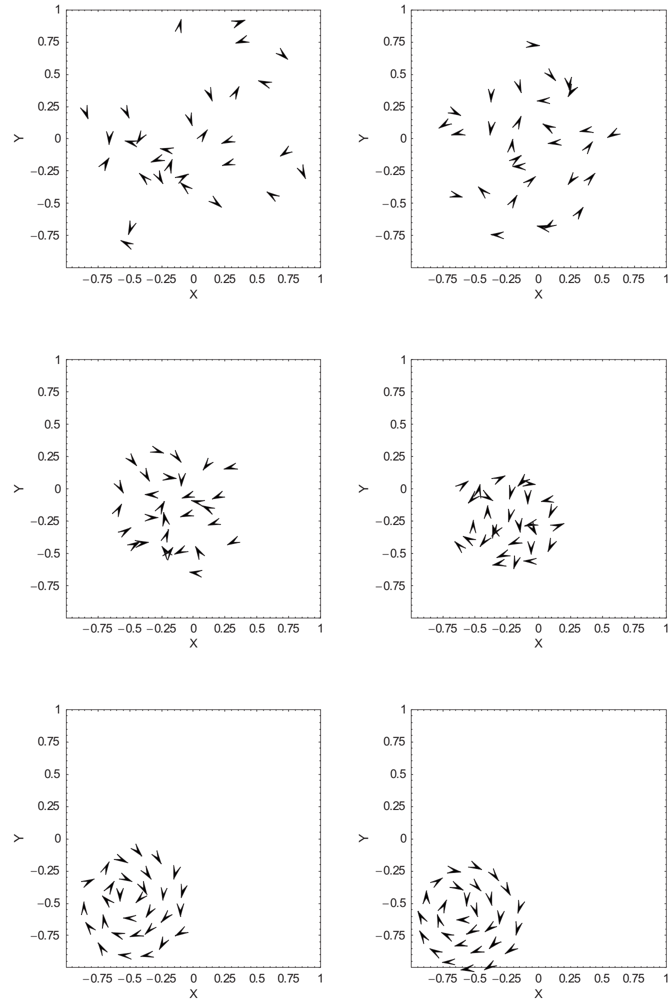

FIG. 2. (Top left to bottom right) Formation of a vortexlike state in a swarm of interacting particles ($N=30$) with $C_a=1$, $C_r=1$, $C_o=0.1$, $l_a=1$, $l_r=0.2$, $l_o=0.5$ for nondimensional time $t=0$ until $t=10$.

from the swarm. An augmented energy function E is now defined as $E=(T+U)-\boldsymbol{\lambda}.(\mathbf{L}-\mathbf{H})$ where $\boldsymbol{\lambda}$ is the Lagrange multiplier which enforces conservation of angular momentum so that

$$E=\left(\frac{1}{2}\sum_i m\mathbf{v}_i^2+\frac{1}{2}\sum_i\sum_j (U_{ij}^a+U_{ij}^r)\right) - \boldsymbol{\lambda}\,.\left(\sum_i m\mathbf{x}_i\times\mathbf{v}_i-\mathbf{H}\right). \tag{2}$$

The total effective energy of the swarm is now minimized by finding a set of states for the swarm particles $(\mathbf{x}_i,\mathbf{v}_i)$ which yield the first variation $\delta E=0$. Differentiating the augmented energy function, the two conditions for $\delta E=0$ are given by

$$\frac{\partial E}{\partial \mathbf{x}_i}=(\boldsymbol{\nabla} U_i^a+\boldsymbol{\nabla} U_i^r)-m\boldsymbol{\lambda}\times\mathbf{v}_i=0, \tag{3a}$$

$$\frac{\partial E}{\partial \mathbf{v}_i} = m(\mathbf{v}_i - \boldsymbol{\lambda} \times \mathbf{x}_i) = 0. \tag{3b}$$

It can be seen immediately from Eq. (3b) that the constrained minimum-energy state of the swarm corresponds to vortex-like rotation. The velocity vector of each particle is normal to its position vector and the vector $\boldsymbol{\lambda}$ such that $\mathbf{v}_i = \boldsymbol{\lambda} \times \mathbf{x}_i$. The Lagrange multiplier $\boldsymbol{\lambda}$ is therefore identified as the angular velocity vector of the swarm which will be directed along $\mathbf{H}$, as shown in Fig. 1. From Eq. (3a) it can be seen that this vortexlike state is achieved when the centripetal acceleration induced by vortexlike rotation is balanced by the gradient of the potential field such that $(\nabla U_i^a + \nabla U_i^r) - m\boldsymbol{\lambda} \times (\boldsymbol{\lambda} \times \mathbf{x}_i) = 0$. It can therefore be concluded that a swarm of particles in an initially random state will relax into spatially coherent vortexlike behavior, as observed in a wide range of biological swarms [1–3] and in simulation [4–7]. It should be noted that setting the first variation $\delta E = 0$ has yielded conditions for E to be stationary, but that a global minimum is not explicitly guaranteed. It may be possible for the swarm to relax to a local constrained minimum-energy state, however it would be expected that fluctuations would lead to escape and relaxation towards the global minimum-energy state.

In order to illustrate the formation of vortexlike structures using the mechanism discussed above, a planar swarm of $N = 30$ particles is considered. The particles in the swarm are randomly distributed over a unit disk with a random distribution of initial velocities such that $|\mathbf{v}_i(0)| < 1$. The free parameters are selected such that $C_a = C_r > C_o$ and $l_r < l_o < l_a$ so that the swarm experiences long-range aggregation, short range repulsion, and local velocity alignment. It can be seen from Fig. 2 that the swarm slowly relaxes to a vortexlike state, while the center-of-mass of the swarm translates with constant velocity, as expected.

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